1) a) $\int \frac{\cos(x)}{1+\sin^2(x)}\,dx = \left[\begin{array}{l} t=\sin(x) \\ dt=\cos(x)\,dx \end{array}\right]$

$= \int \frac{1}{1+t^2}\,dt = \arctan(t)+C$

$= \arctan(\sin(x))+C.$

b) $\int \frac{4x^2}{\sqrt{x^3+2}}\,dx = \left[\begin{array}{l} t=\sqrt{x^3+2} \\ dt=\frac{1}{2}(x^3+2)^{-1/2}\cdot 3x^2\,dx \end{array}\right]$

$= \int \frac{8}{3}\,dt = \frac{8}{3}t+C = \frac{8}{3}\sqrt{x^3+2}+C.$

c) $\int_{\pi/4}^{\pi/3} \frac{\sin(x)}{\cos(x)}\,dx = \left[\begin{array}{l} t=\cos(x) \\ dt=-\sin(x)\,dx \\ x=\frac{\pi}{4} \Rightarrow t=\cos(\frac{\pi}{4})=\frac{1}{\sqrt{2}} \\ x=\frac{\pi}{3} \Rightarrow t=\cos(\frac{\pi}{3})=\frac{1}{2} \end{array}\right]$

$= \int_{1/\sqrt{2}}^{1/2} -\frac{dt}{t} = \int_{1/2}^{1/\sqrt{2}} \frac{dt}{t} = \Big[\ln|t|\Big]_{1/2}^{1/\sqrt{2}} = \ln\left(\frac{1}{\sqrt{2}}\right)-\ln\left(\frac{1}{2}\right)$

$= \ln\left(\frac{1/\sqrt{2}}{1/2}\right) = \ln\left(\frac{2}{\sqrt{2}}\right) = \ln(\sqrt{2}) = \frac{1}{2}\ln(2).$

Eller: Notera att $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

$\Rightarrow \int_{\pi/4}^{\pi/3} \frac{\sin(x)}{\cos(x)}\,dx = -\int_{\pi/4}^{\pi/3} \frac{-\sin(x)}{\cos(x)}\,dx = \Big[-\ln|\cos(x)|\Big]_{\pi/4}^{\pi/3}$

$= -\ln\left(\frac{1}{2}\right)-\left(-\ln\left(\frac{1}{\sqrt{2}}\right)\right) = \ln\left(\frac{1}{\sqrt{2}}\right)-\ln\left(\frac{1}{2}\right)$

$= [\text{se ovan}] = \frac{1}{2}\ln(2).$

2)

a) $V = \int_1^e \pi y^2 \, dx = \int_1^e \pi x^2 \ln(x) \, dx$

$$= \left[ \begin{array}{l} \text{Partialintegration:} \\ \quad f(x) = x^2, \quad g(x) = \ln(x) \\ \quad F(x) = \frac{x^3}{3}, \quad g'(x) = \frac{1}{x} \end{array} \right]$$

$$= \pi \left( \left[ \frac{x^3}{3} \cdot \ln(x) \right]_1^e - \int_1^e \frac{x^3}{3} \cdot \frac{1}{x} \, dx \right)$$

$$= \pi \left( \frac{e^3}{3} - \int_1^e \frac{x^2}{3} dx \right)$$

$$= \pi \left( \frac{e^3}{3} - \left[ \frac{x^3}{9} \right]_1^e \right)$$

$$= \pi \left( \frac{e^3}{3} - \left( \frac{e^3}{9} - \frac{1}{9} \right) \right)$$

$$= \pi \cdot \frac{2e^3+1}{9} .$$

b)

$$\begin{cases} \ln(1-2x) = -2x - \frac{(-2x)^2}{2} + x^3 B_1(x) = -2x - 2x^2 + x^3 B_1(x) \\ \arctan(4x) = 4x + x^3 B_2(x) \end{cases}$$

<u>Detta ger:</u>

$$\frac{\ln(1-2x) - 2x + \arctan(4x)}{x^2} = \frac{-2x - 2x^2 + x^3 B_1(x) - 2x + 4x + x^3 B_2(x)}{x^2}$$

$$= \frac{-2x^2 + x^3 B_3(x)}{x^2} = -2 + x B_3(x) \longrightarrow \underline{\underline{-2}}, \text{ då } x \to 0 .$$

Här är alla $B_i(x)$ begr. funktioner nära 0.

3) a) $y(x) = 3x^2 - \int_0^x 2t\,y(t)\,dt$

Notera att $\underline{\underline{y(0)}} = 0 - \int_0^0 2t\,y(t)\,dt = \underline{\underline{0}}$.

Derivera och använd Analysens huvudsats:

$y'(x) = 6x - 2xy \iff y' + 2xy = 6x$

$IF = e^{\int g(x)dx} = e^{\int 2x dx} = e^{x^2}$.

Dvs,
$$(y \cdot e^{x^2})' = 6x e^{x^2}$$
$$\iff y e^{x^2} = \int 6x e^{x^2} dx = 3e^{x^2} + C$$
$$\iff y = 3 + Ce^{-x^2}$$

$y(0) = 0$ ger: $0 = 3 + C \iff C = -3$.

Dvs, $y(x) = 3 \cdot (1 - e^{-x^2})$.

b) $y' = x(1+y^2)$ ; $y(0) = 1$.

Separera variabler:

$$\frac{dy}{1+y^2} = x\,dx \iff \int \frac{dy}{1+y^2} = \int x\,dx$$

$$\iff \arctan(y) = \frac{x^2}{2} + C$$

$y(0) = 1$ ger: $\arctan(1) = \frac{\pi}{4} = C$

Dvs, $y = \tan\left(\frac{x^2}{2} + \frac{\pi}{4}\right)$.

4) a) Integralen är generaliserad i $x=1$.

Beräkna integralens värde (om konv.):

$$\lim_{\varepsilon\to 0^+} \int_{1+\varepsilon}^{5} \frac{1}{\sqrt{x-1}}\,dx = \lim_{\varepsilon\to 0^+} \int_{1+\varepsilon}^{5} (x-1)^{-1/2}\,dx$$

$$= \lim_{\varepsilon\to 0^+} \left[\frac{(x-1)^{1/2}}{1/2}\right]_{1+\varepsilon}^{5} = \lim_{\varepsilon\to 0^+} 2\left(\sqrt{4}-\sqrt{\varepsilon}\right) = 2\sqrt{4} = 4.$$

Alltså är integralen konvergent med värdet 4.

b) Börja med polynomdivision:

$$\begin{array}{r|l} & x^3+x^2 \\ \hline x^2-x & x^5-x^3+1 \\ & x^5-x^4 \\ \hline & x^4-x^3+1 \\ & x^4-x^3 \\ \hline & 1 \end{array}$$

Ger att: $$\frac{x^5-x^3+1}{x^2-x} = x^3+x^2+\frac{1}{x^2-x} = x^3+x^2+\frac{1}{x(x-1)}$$

Partialbråksuppdelning: $$\frac{1}{x(x-1)} = \frac{A}{x}+\frac{B}{x-1}$$

$\Rightarrow\ 1 = A\cdot(x-1)+B\cdot x.$

Ger: $\begin{cases} 1=-A \\ 0=A+B \end{cases} \Leftrightarrow \begin{cases} A=-1 \\ B=1 \end{cases}$. Dvs, $\dfrac{x^5-x^3+1}{x^2-x} = x^3+x^2+\dfrac{1}{x-1}-\dfrac{1}{x}$

$$\Rightarrow \int \frac{x^5-x^3+1}{x^2-x}\,dx = \int\left(x^3+x^2+\frac{1}{x-1}-\frac{1}{x}\right)dx = \frac{x^4}{4}+\frac{x^3}{3}+\ln|x-1|-\ln|x|+C$$

$$= \frac{x^4}{4}+\frac{x^3}{3}+\ln\left|\frac{x-1}{x}\right|+C.$$

5) $y'' - 2y' + y = \sin(2x)$

Homogen lösning:

Kar. ekv: $r^2 - 2r + 1 = 0 \Leftrightarrow (r-1)^2 = 0 \Leftrightarrow r = 1$ (dubbelrot)

$\Rightarrow y_h = (A + Bx)e^x$.

Partikulär lösning:

Hjälpekv: $u'' - 2u' + u = e^{i2x}$ (Notera: $\sin(2x) = \operatorname{Im} e^{i2x}$)

Ansats: $u = z \cdot e^{i2x}$

$u' = e^{i2x} \cdot (z' + 2iz)$

$u'' = e^{i2x} \cdot (z'' + 4iz' - 4z)$

$\Rightarrow z'' + 4iz - 4z - 2 \cdot (z' + 2iz) + z = 1$

$\Leftrightarrow z'' + (-2 + 4i) z' + (-3 - 4i) z = 1$

Ansats: Sätt $z = C$ (= konstant) $\Rightarrow z' = z'' = 0$

Dvs, $(-3 - 4i) \cdot C = 1 \Rightarrow z = \dfrac{1}{-3-4i} = -\dfrac{3-4i}{25}$.

$\Rightarrow u = \left(-\frac{3}{25} + \frac{4}{25}i\right) \cdot e^{i2x} = \left(-\frac{3}{25} + \frac{4}{25}i\right) \cdot \left(\cos(2x) + i\sin(2x)\right)$

$= -\frac{3}{25} \cdot \cos(2x) - \frac{3}{25} i \sin(2x) + \frac{4}{25} i \cos(2x) - \frac{4}{25} \sin(2x)$

Detta ger: $y_p = \operatorname{Im}(u) = -\frac{3}{25}\sin(2x) + \frac{4}{25}\cos(2x)$

Totalt: $y = y_h + y_p = (A + Bx)e^x - \frac{3}{25}\sin(2x) + \frac{4}{25}\cos(2x)$.

6) a) Behållare 1:

Def: $x(t)$ = saltmängd vid tid $t$

kg/l

$$x'(t) = x_{in} - x_{ut} = 0\cdot 3 - \frac{x}{100}\cdot 3 \quad [kg/min]$$

l/min

$$\Rightarrow x' = -\frac{3}{100}\cdot x \iff x' + 0{,}03\cdot x = 0$$

$$\iff x = C\cdot e^{-0{,}03t} \quad \text{(Lös med t.ex. IF)}$$

$x(0) = 10$ ger: $10 = C\cdot e^0 \iff C = 10$.

$$\boxed{\text{Alltså: } x(t) = 10\cdot e^{-0{,}03\cdot t}}$$

Behållare 2:

Def: $y(t)$ = saltmängd (i behållare 2) vid tid $t$

$$y'(t) = y_{in} - y_{ut} = \frac{x}{100}\cdot 3 - \frac{y}{100}\cdot 3$$

$$\Rightarrow y' = 0{,}03\cdot x - 0{,}03\cdot y$$

$$= 0{,}03\cdot 10\cdot e^{-0{,}03t} - 0{,}03\cdot y$$

$$\iff y' + 0{,}03y = 0{,}3e^{-0{,}03t} \iff y = e^{-0{,}03t}\cdot(C_1 + 0{,}3t)$$

(Lös med t.ex. IF).

$y(0) = 0$ ger $0 = e^0\cdot(C_1 + 0) \iff C_1 = 0$

Dvs, $y = 0{,}3te^{-0{,}03t}$.

Antag att $x(t) = y(t)$ vid tid $t$, dvs,

$$10e^{-0{,}03t} = 0{,}3\cdot te^{-0{,}03t} \iff 10 = 0{,}3t \iff t = \frac{10}{0{,}3}$$

Dvs, vid tid $t = \frac{10}{0.3}$ innehåller varje behållare $\frac{10}{e}$ kg salt.