Turbulence – Theory and Modelling

Course information and short introduction

By Jörgen Held
Lund, 1999

Updated by Daniel Lörstad (2000) and Johan Revstedt (2005, 2006, 2007)

Schedule 2009
Turbulence - Theory and Modeling MVK140/MMV010F

Date	*Room*	*What/Who*	*Chapter in Pope*	*Subject*
Week 1				
26/10 13.15-15.00	M:M2	L-JR	Ch. 1	Introduction
27/10 13.15-15.00	E:1407	L-JR	Ch. 2, App. A-B	Tensor calculus + basic eq.s
29/10 13.15-15.00	M:2314	Lab-RS/JR	Flow instability	Taylor-Couette flow
29/10 15.15-17.00	M:2314	Lab-RS/JR	Flow instability	Taylor-Couette flow
30/10 08.15-10.00	E:1405	L-JR	Ch. 3-4	Statistical tools, Averaged eqs.
Week 2				
2/11 13.15-15.00	M:M2	L-JR	handouts	Transition
4/11 13.15-12.00	E:1407	L-JR	Ch. 5	Free shear flows
5/11 13.15-15.00	M:L2	L-JR	Ch. 5,7	Free shear flows cont. + wall bounded flows
5/11 15.15-17.00	M:L2	P- RS	Ch. 1-4	
6/11 08.15-10.00	E:1405	E- RS	Ch. 1-4	
Week 3				
9/11 13.15-15.00	M:M2	L-JR	Ch. 7	Wall bounded flows cont.
11/11 13.15-15.00	E:1407	L-JR	Ch. 6	Scales of turbulent flow
12/11 13.15-15.00	M:L2	P-RS	Ch. 5-7	
12/11 15.15-17.00	M:L2	E-RS	Ch. 1-7	
13/11 08.15-10.00	E:1405	L-RS		Introduction to OpenFOAM
Week 4				
16/11 13.15-15.00	M:M2	L-JR	Ch. 8,9	Introduction to turb.modeling+ DNS
17/11 13.15-15.00	E:1407	L-JR	Ch. 10	Two-Equation models
19/11 13.15-15.00	TBA	Lab-JR/RS	Ch. 11	Computer exercise 1
19/11 15.15-17.00	TBA	Lab-JR/RS		Computer exercise 1
20/11 08.15-10.00	E:1405	P/E-RS	Ch. 8,10	Exercise 3
Week 5				
23/11 13.15-15.00	M:M2	L-JR	Ch. 10	Two-Equation models cont.
24/11 13.15-15.00	E:1407	L-JR	Ch. 11	Reynolds stress models
26/11 13.15-15.00	TBA	Lab-JR/RS		Computer exercise 2
26/11 15.15-17.00	TBA	Lab-JR/RS		Computer exercise 2
27/11 08.15-10.00	E:1405	P/E-RS	Ch. 8-11	Exercise 3b
Week 6				
30/11 13.15-15.00	M:M2	L	Turbulence in plasmas	Guest lecture
1/12 13.15-15.00	E:1407	E-JR		
3/12 13.15-15.00	M:L2	L-RS	Ch. 13	LES
3/12 15.15-17.00	M:L2	P-RS	Ch. 11,13	
Week 7				
8/12 13.15-15.00	E:1407	S-JR/RS		PhD student seminars
10/12 13.15-15.00	M:L2	G-JR/RS	GS2	Summary & Presentation GS2
10/12 15.15-17.00	M:L2	L		Guest lecture, TetraPak

L = Lecture
GS = Group study
E= Exercise
P= Problem solving seminar
S= Seminar
Lab = Laboratory exercise

JR = Johan Revstedt
RS = Robert Szasz

Contents

I. Information
   - Goals
   - Course requirements

II. Introduction to transition

III. Introduction to turbulence

IV. Introduction to turbulence modelling

V. Problems

VI. Answer to problems

VII. Terminology

Appendix about tensor notation

I. Information

Goals:

Knowledge and Understanding

For a passing grade the student must:

- be able to describe the physical mechanisms of the transition from laminar to turbulent flow for a simple flow case
- be able to explain Kolmogorov's theory, including the basic assumptions and the validity of the theory
- be able to, from a phenomenological perspective, assess if a flow is turbulent
- be able to explain some of the important and basic terms of the subject
- be able to describe the character of the turbulence in different flow situations with respect to the properties and development of the turbulence, and explain how the differences between these flow situations are reflected in the modelling

Skills and Abilities

For a passing grade the student must:

- be able to analyse a flow case and suggest a method for numerical simulation with respect to governing equations, possible simplifications and choice of turbulence model, and also to compare with alternative methods.
- be able to scrutinise and from given criteria estimate the credibility of results from turbulent flow simulations

Values and Attitudes

For a passing grade the student must:

- be able to actively participate in discussion of problems relevant for the subject
- be able to present, both orally and in writing, a technical report containing analyses and choice of turbulence model

Course requirements

The course is based on lectures, exercises and laboratory exercises. In addition there are two compulsory home works, one for the turbulence theory part (GS1) and one for the modelling part (GS2). Written reports of these home works should be handed in latest **17 November 5**

pm (GS1) and **7 December 8 am (GS2).** The reports should be approved before the exam. In addition GS2 shall also be presented orally **(10 min)** on **10 December**. Each group will also get the report of another group before 5 pm on 7 December and should prepare a few (at least three) questions on that report.
The exam is oral and will be held according to a separate schedule. Written reports of the laboratory exercises must also be handed in. Note that there will be guest lectures on **30 November** and **10 December**.

Course literature:

Pope, S.B., Turbulent Flows, Cambridge University Press, 2000
NB. Chapters 6.2, 6.4 and 12 are not included.

Courses on the Web

Information, handouts etc will be posted on the course web-page. The address is http://www.fm.vok.lth.se/Courses/MVK140/turbmod_uk.htm.

Notation in this folder

The notation for instantaneous, averaged and fluctuating properties are quite different from book to book.

Landahl & Mollo-Christensen:

$$U_i(velocity), \quad \overline{U}_i(averaged\ velocity)\ and\ u_i(fluctuating\ velocity)$$

Tennekes & Lumley:

$$\tilde{u}_i(velocity), \quad U_i(averaged\ velocity)\ and\ u_i(fluctuating\ velocity)$$

Wilcox:

$$u_i(velocity), \quad U_i(averaged\ velocity)\ and\ u_i'(fluctuating\ velocity)$$

Pope:

$$U_i(velocity), \quad \langle U_i \rangle(averaged\ velocity)\ and\ u_i(fluctuating\ velocity)$$

In the problems (and answers) in this folder the notation by Wilcox is used.

II. Introduction to transition

Laminar flow tend to become unstable at high Reynolds number. Instability to small disturbances is an initial step whereby a laminar flow goes through transition to turbulence. Transition has been extensively studied, but many of the underlying mechanisms are still not fully understood. Main contributions have been:

- Rayleigh (1878) proved the inflection point theorem stating that a necessary condition for inviscid instability is that the mean velocity profile has an inflection point.
- Orr (1907) and Sommerfeld (1908) derived independently the governing viscous stability equation by applying the normal mode analysis to the Navier-Stokes equations.
- Solutions to these equation were first obtained by Tollmien (1929) when he calculated the first neutral eigenvalues and obtained a critical Reynolds number for the flat plate boundary layer.
- The theoretical knowledge of boundary layer stability was further enhanced by Schlichting (1930-) determining the eigenfunctions of the oscillation and the degree of amplification of the disturbances that arise.
- Squire (1933) pointed out that only two dimensional waves need to be considered (two dimensional waves become unstable first).
- The two dimensional Tollmien-Schlichting wave that should appear in the boundary layer before transition was experimentally confirmed by Schubauer & Skramstad (1947)

For boundary layer flows a general picture of the disturbance development from an initial stage to transition can be drawn. The initial amplification of the disturbance is in close agreement to what is predicted for linear Tollmien-Schlichting waves but the waves become three dimensional and start to bend into regularly spaced hairpins or U-loops (See p. 110 in Landahl & Mollo-Christensen). The top of the loop is lifted from the plate and stretched by the streamwise velocity variation normal to the wall. As the vortex tube is stretched the local pertubation velocity at the top is increased.

III. Introduction to turbulence

Most flows encountered in nature and in engineering practice are turbulent. In fluid dynamics laminar flow is the exception, not the rule. It is difficult to give a precise definition of turbulence. All one can do is to list some of the characteristics of turbulent flows (Tennekes & Lumley).

- Irregularity: The irregularity or randomness is one characteristic of all turbulent flows. This make a deterministic approach impossible. Instead one relies on statistical methods.
- Diffusivity: The diffusivity of turbulence causes rapid mixing and increased rates of momentum, heat and mass transfer. The diffusivity of turbulence prevents boundary-layer separation on wings at large angles of attack, increase the wall friction and increase heat transfer and mixing in machinery of all kinds.
- Large Reynolds number. Turbulent flows always occur at high Reynolds numbers. Turbulence often originates as an instability of laminar flows if the Reynolds number is large enough. The instabilities are related to the

interaction of viscous terms and non-linear inertia terms. This interaction is very complex.

- Three-dimensional vorticity fluctuations: Turbulence is rotational and three dimensional. The random vorticity fluctuations that characterise turbulence could not maintain themselves if the velocity fluctuations were two dimensional since an important vorticity-maintenance mechanism due to vortex stretching is absent in two dimensional flows.
- Dissipation: Turbulent flows are always dissipative. Viscous shear stresses perform deformation work at the expense of kinetic energy of the turbulence. Turbulence needs a continuous supply of energy to make up for the viscous losses. If no energy is supplied, turbulence decays ultimately.
- Continuum. Turbulence is a continuum phenomenon, governed by the equation of fluid mechanics. Even the smallest scales occuring in a turbulent flow are ordinarily far larger than any molecular length scale (mean free path).
- Turbulence is a property of the flow. Turbulence is not a feature of fluids but of fluid flows. Most of the dynamics of turbulence is the same in all fluids, whether they are liquids or gases.

In turbulent flows all scales between the largest and the smallest are present. This will be demonstrated at the "Reynolds experiment"-laboration in the course. Different characteristic scales frequently used in the field of turbulence are: The integral scale, the Taylor micro-scale and Kolmogorov micro-scale.

IV. Introduction to turbulence modelling

The most general description of a fluid is obtained from the full system of Navier-Stokes equations. Analytical solutions to these equations exists only for some simplified low Reynolds number flows. Numerical solutions are sought instead. It is impossible to resolve all scales in a high Reynolds number flow even if the largest/fastest supercomputer of today is used. One has to rely on a statistical description of turbulence. One concept widely used is the Reynolds decomposition of the flow variables. The variables are decomposed into a mean part and a fluctuating part.

$$u_i = U_i + u_i'$$

Inserting the Reynolds decomposed variables in the governing equations for incompressible flow gives rise to an unknown term, the Reynolds stress tensor.

$$\rho \frac{\partial \overline{u_i u_j}}{\partial x_j} = \rho \frac{\partial \overline{u_i}\,\overline{u_j}}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}$$

The problem is how to model this term in an accurate way. Boussinesq (1877) proposed that the turbulent stress could be expressed as a turbulent viscosity times the mean rate of strain.

$$\tau_{ij} = -\frac{2}{3}\rho k \delta_{ij} + 2\mu_T S_{ij}$$

In analogy with molecular viscosity (even though turbulent viscosity is a property of the flow and molecular viscosity is a property of the fluid) one tried to express the turbulent viscosity in terms of a characteristic velocity and a characteristic length.
Prandtl (1925) put forward his mixing length model. He replaced v_{mix} with l_{mix} times a

$$\mu_T \approx \rho v_{mix} l_{mix}$$

velocity-gradient. Now, l_{mix} is the only unknown that has to be determined.

Kolmogorov (1942) proposed a transport equation for the turbulent kinetic energy and for the specific dissipation rate which based on dimensional analysis provide a characteristic velocity and a characteristic length.

Jones and Launder (1972) introduced the k-ε model often referred to as the Standard k-ε model. These models are used only to determine a characteristic velocity and a characteristic length used in the eddy-viscosity concept.

$$v_{mix} = \sqrt{k}$$

$$l_{mix} = \frac{k^{3/2}}{\varepsilon}$$

Two-equations models (such as the k-ε model and the k-ω model) are still dominating in the context of industrial flow computations and provide excellent predictions for many flows of engineering interest. However, there are some applications for which predicted flow properties differ greatly from corresponding measurements. Some of the most noteworthy types of applications for which models based on the Boussinesq approximation fail are:

- Flow over curved surfaces. Many practical aerodynamic surfaces are sufficiently curved to produce significant curvature effects.
- Separated flows, especially when the flow is compressible.
- Flow in non-circular ducts (secondary motion).
- Flows with sudden changes in mean rate of strain.

The above mentioned deficits may be circumvented by modelling the Reynolds stress tensor directly through some algebraic relation or through a transport equation for the Reynolds stress tensor resulting in what is called Reynolds stress models.
The exact differential equations (for incompressible flow) describing the behaviour of the Reynolds stress tensor reads

$$\frac{\partial \overline{u_i' u_j'}}{\partial t} + \overline{u}_k \frac{\partial \overline{u_i' u_j'}}{\partial x_k} = P_{ij} - \varepsilon_{ij} + \Pi_{ij} + D_{ij}$$

Where P_{ij} is the Reynolds stress production tensor. Energy is transferred from the mean flow to the turbulent scales. The shear in the mean flow is a source of energy for turbulent velocity fluctuations, ε_{ij} is the dissipation rate tensor, Π_{ij} is the pressure-strain rate tensor and D_{ij} is the turbulent transport tensor. The tensor can be seen as a tensor describing the turbulent diffusion. See Wilcox p.17-19 for more details.

A totally different approach is applied in Large Eddy Simulations. Here the flow variables are decomposed into a resolved (filtered) and an unresolved (sub-filter) part. Because the small (sub-filter) scales are very universal in character they are much more amenable to general modelling.

V Problems

1.1 List the assumptions made in deriving the Orr-Sommerfeld equation.
1.2 What will happen with the disturbances if $c_i^{(n)} > 0$
1.3 What is meant by a "critical" Reynolds number?
1.4 In which way is a velocity profile with an inflection point related to transition?
1.5 How will a favourable pressure gradient affect transition for a flow past a flat plate?

2.1 Describe and explain the characteristics of turbulence.

2.2 Which variables make up the Reynolds number? Give their dimensions. The Reynolds number describes a ratio between two different terms in the flow equations. Which terms?

2.3 Why and from where arises turbulence?

2.4 Energy supply is needed to maintain the turbulence. Give two sources which provide the turbulence with energy.

2.5 Give the dimension of the dissipation rate, ε.

2.6 To estimate Kolmogorov micro-scales of length, time and velocity, we assume that they depend on only two variables. Which variables?

2.7 Use the two variables in prob. 2.6 to estimate Kolmogorov micro-scales of length, time and velocity by dimensional analysis.

2.8 Form a Reynolds number using Kolmogorov micro-scales. Which value is obtained?

2.9 Express the rate of dissipation, ε, using variables related to large scale turbulence.

2.10 Use the results from prob. 2.7 and 2.9 to estimate the scale relation (length, time and velocity) between small scale and large scale turbulence expressed in form of a suitable Reynolds number.

3.1 Write out all elements in the following expression, where a is a constant and δ_{ij} is the Kronecker delta.

$$B_{ij} = a\delta_{ij}$$

3.2 The tensor C_{ij}, shown below, contains 3x3 elements.

$$C_{ij} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

Show that the contraction of the tensor C_{ij} can be written as:

$$C_{ij}\delta_{ji} = c_{ii}$$

3.3 Write out all the terms in the following expressions.

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad \text{Continuity}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} \qquad \text{Momentum}$$

$$\sigma_{ij} = -p\delta_{ij} + 2\mu S_{ij} \qquad \text{Molecular stress tensor with pressure term}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \text{Rate of strain}$$

3.4 Obtain the averaged momentum equations using the instantaneous continuity and the momentum equations.

3.5 Derive the transport equation for Reynolds stresses.

3.6 Give the number of independent terms in Reynolds stress tensor.

3.7 Define the Knudsen number. In what way is the Knudsen number related to the assumption of a continuum?

3.8 Explain why eddies, whose principal axis roughly aligned with that of mean strain rate, are effective in extracting energy from the mean flow.

3.9 Define the friction velocity, u_*

4.1 Derive the transport eq. for the kinetic energy in terms of mean flow variables.

4.2 Derive the transport eq. for the turbulent kinetic energy.

4.3 Explain the different terms in the derived equation in prob. 4.2.

4.4 Define the Taylor micro-scale.

4.5 At which scales is energy from the mean flow supplied to the turbulence and at which scales is the turbulent kinetic energy dissipated?

5.1 Consider the flow between two long concentric cylinders, shown at the demonstration earlier in the course. Calculate at which angular velocity the critical Taylor number occurs.

$$\mathrm{Ta}_{\mathrm{crit}} = \frac{\mathrm{r_i}(\mathrm{r_o} - \mathrm{r_i})^3 \Omega_\mathrm{i}^{\,2}}{\nu^2} \approx 1700$$

$$\mathrm{r_i} = 25\mathrm{mm}, \quad \mathrm{r_o} = 37{,}5\mathrm{mm}, \quad \nu = 14 \cdot 10^{-6}\,\mathrm{m}^2/\mathrm{s}$$

5.2 Consider a flow through a pipe. How is the friction affected by turbulence? Motivate.

5.3 Consider a laminar flow through a bent pipe. The pressure drop is related to the velocity squared. Explain why the pressure drop increases over the bend.

6.1 Why do we need turbulence models?

6.2 Explain why time- or ensemble-averaged turbulence models lack in universality.

6.3 Why can eddy-viscosity based turbulence models not predict the turbulence accurately when there is a sudden change in the mean rate of strain?

6.4 What is the purpose of deriving transport equations for unknown correlation terms when they in turn will contain higher correlation terms (i.e. the closure problem)?

VI. Answer to problems

1.1 Parallel shear flow
Constant density
Infinitesimal perturbation in the normal direction
Terms, quadratic in perturbation quantities, are neglected

1.2 The amplitude of the perturbation/disturbance will grow with time at a rate of $e^{\alpha c i t}$. Note that the theory predicts only the initial growth of the perturbation/disturbance (i.e. t is small)

1.3 The critical Reynolds number indicates that the flow will be at the critical stage. Beyond this critical number the flow will become unstable (disturbances will be amplified). Below this value, disturbances will decay.

1.4 A necessary condition for instability is that the velocity profile possesses a point of inflection.

1.5 A favourable pressure gradient tend to stabilise the flow past a flat plate. The velocity profile will be "fuller" and it will be more difficult for the flow to develop an inflection point.

2.1 See introduction to turbulence in this folder.
Irregularity, Diffusivity, Large Reynolds number, Three-dimensional vorticity fluctuations, Dissipative, Continuum and Turbulent flows are flows

2.2 Density [kg/m^3], velocity [m/s], length [m] and viscosity [kg/sm]
Alt. Velocity [m/s], length [m] and kinematic viscosity [m^2/s]
The Reynolds number describes the ratio between convection terms and the molecular diffusion terms

2.3 Turbulence arises from instabilities at large Reynolds numbers.

2.4 From shear and buoyancy.

2.5 ε [m^2/s^3]

2.6 Dissipation rate and kinematic viscosity.

2.7 Kolmogorov micro-scale

Length $\eta \equiv (\nu^3/\varepsilon)^{1/4}$
Time $\tau \equiv (\nu/\varepsilon)^{1/2}$
Velocity $v \equiv (\nu\varepsilon)^{1/4}$

2.8 $$Re = \frac{\eta v}{\nu} = 1$$

2.9 $$\varepsilon \sim \frac{u^3_{RMS}}{l}$$

2.10 $$\frac{\eta}{l} \sim \frac{(\nu^3/\varepsilon)^{1/4}}{l} = \frac{(\nu^3/(u^3_{RMS}/l))^{1/4}}{l} = \left(\frac{u_{RMS} l}{\nu}\right)^{-3/4} = Re_l^{-3/4}$$

$$\frac{\tau}{t} \sim \frac{\tau u_{RMS}}{l} = \ldots = Re_l^{-1/2}$$

$$\frac{v}{u_{RMS}} \sim \ldots = Re_l^{-1/4}$$

3.1 3.2 $$B_{ij} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B33 \end{pmatrix} = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

$$C_{ij}\delta_{ji} = c_{11}\cdot 1 + c_{12}\cdot 0 + c_{13}\cdot 0 + c_{21}\cdot 0 + c_{22}\cdot 1 + c_{23}\cdot 0 +$$
$$c_{31}\cdot 0 + c_{32}\cdot 0 + c_{33}\cdot 1 = c_{11} + c_{22} + c_{33} = C_{ii}$$

3.3 $$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = \frac{1}{\rho}\left(\frac{\partial(-p + 2\mu s_{xx})}{\partial x} + \frac{\partial(2\mu s_{xy})}{\partial y} + \frac{\partial(2\mu s_{xz})}{\partial z}\right)$$

$$\frac{\partial v}{\partial t} + \frac{\partial vu}{\partial x} + \frac{\partial vv}{\partial y} + \frac{\partial vw}{\partial z} = \frac{1}{\rho}\left(\frac{\partial(2\mu s_{yx})}{\partial x} + \frac{\partial(-p + 2\mu s_{yy})}{\partial y} + \frac{\partial(2\mu s_{yz})}{\partial z}\right)$$

$$\frac{\partial w}{\partial t} + \frac{\partial wu}{\partial x} + \frac{\partial wv}{\partial y} + \frac{\partial ww}{\partial z} = \frac{1}{\rho}\left(\frac{\partial(2\mu s_{zx})}{\partial x} + \frac{\partial(2\mu s_{zy})}{\partial y} + \frac{\partial(-p + 2\mu s_{zz})}{\partial z}\right)$$

$$\sigma_{ij} = -\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + 2\mu\begin{pmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{yx} & s_{yy} & s_{yz} \\ s_{zx} & s_{zy} & s_{zz} \end{pmatrix}$$

$$S_{ij} = \frac{1}{2}\begin{pmatrix} (\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}) & (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \\ (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) & (\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}) & (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) \\ (\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}) & (\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}) & (\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}) \end{pmatrix}$$

3.4 Use the product rule for partial derivatives. The rightmost term on the right hand side is identical to zero (continuity eq.)

$$\rho\left(\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j}\right) = \rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial u_j}{\partial x_j}\right)$$

3.5 $$\rho\left(\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial (2\mu s_{ij})}{\partial x_j}$$

Put in Reynolds decomposed variables

$$\rho\left(\frac{\partial (U_i + u_i')}{\partial t} + \frac{\partial ((U_i + u_i') + (U_j + u_j'))}{\partial x_j}\right) = -\frac{\partial (P + p')}{\partial x_i} + \frac{\partial (2\mu (S_{ij} + s_{ij}'))}{\partial x_j}$$

Perform the time averaging

$$\rho\left(\frac{\partial U_i}{\partial t} + \frac{\partial (U_i U_j + \overline{u_i' u_j'})}{\partial x_j}\right) = -\frac{\partial P}{\partial x_i} + \frac{\partial (2\mu S_{ij})}{\partial x_j}$$

3.6 Reynolds stress tensor is symmetric containing six independent terms.

3.7 The Knudsen number is the ratio between the mean free path for atoms and the Kolmogorov micro-scale. For the assumption of a continuum to be valid, the Knudsen number has to be much less than one.

3.8 Eddies, whose principal axis is aligned with that of the mean strain rate, will be stretched. The eddy radius will decrease because of mass conservation. Conservation of angular momentum will in turn increase the angular velocity quadratically.

$$H = mr^2\omega \qquad \text{Angular momentum}$$

This leads to an energy transport from the mean flow to the eddies because the eddy energy will increase when the radius decreases and the angular velocity increases quadratically.

$$m\frac{u^2}{2} = m\frac{(r\omega)^2}{2} \qquad \text{Eddy energy}$$

When the angular velocity increases smaller eddies lying perpen-dicular to the stretched eddy will in turn be stretched.

3.9 $\tau_{wall} = \rho u_*^2$

4.1 Multiply the momentum equations with U_i perform the averaging and make use of the following relation

$$U_i U_j \frac{\partial U_i}{\partial x_j} = U_j \frac{\partial \frac{1}{2} U_i U_i}{\partial x_j} = U_j \frac{\partial K}{\partial x_j}$$

4.2/4.3 See chap. 2.4 p. 19 and chap. 4.1 p. 74 in Wilcox.

$\tau_{ij} \frac{\partial U_i}{\partial x_j}$ production of turb. kin. energy

$\rho\varepsilon$ dissipation of turb. kin. energy

$\frac{\partial}{\partial x_j}\left(\mu \frac{\partial k}{\partial x_j}\right)$ diffusion of turb. kin. energy

$\frac{\partial}{\partial x_j}(\frac{1}{2}\overline{u_i' u_i' u_j'})$ turb. transport of turb. kin. energy

$\frac{\partial}{\partial x_j}(\overline{p' u_i'})$ pressure diffusion term

4.4

4.5 Transfer of energy from the mean flow to the turbulence occurs mainly at the large scales, (integral scale l) and viscous dissipation occurs mainly at scales comparable to Kolmogorov micro-scale, η.

5.1 2.61[rad/s]

5.2 The friction is increased by the presence of turbulence. The velocity gradient at the wall is larger for a turbulent flow than for a laminar flow due to the fact that turbulence brings high speed flow to the wall.

5.3 The velocity at the outer surface of the bend will increase and the velocity at the inner surface decrease. The pressure drop (due to friction) which is proportional to the velocity squared will increase much more at the outer surface than it decreases at the inner surface. The net effect is often taken into account, by engineers, through a friction coefficient, ζ, times the velocity squared when pipe flow friction is to be calculated.

6.1 Due to limited computer capacity of today one has to treat the turbulence in a statistical manner. Averaging gives rise to an unknown correlation term. This term has to be modelled.

6.2 The model should account for all fluctuating scales influence on the mean variables. The large scales are dependent on the boundary conditions for the considered flow. (The models have to be calibrated for different flows).

6.3 Sudden change in the mean rate of strain implies sudden change in turbulent kinetic energy if the eddy-viscosity concept is applied. This is not in accordance with the physical behaviour of turbulence.

6.4 Properties in the transport equations sometimes have a physical meaning and are consequently easier to model or can be obtained by experimental measurements,

however additional equations will increase the computational work and memory storage.

Review: There exists no analytical solution to turbulent flows. The supercomputers of today are not able to perform a Direct Numerical Simulation of a turbulent flow at moderate/high Reynolds number. We have to rely on a statistical description of the turbulent flow. There are two options. We can solve the mean flow field (Reynolds Averaged Navier-Stokes Simulation, RANS) and model the correlation terms that arise when we derive the mean flow equations or resolve the large scales (Large Eddy Simulation, LES) and model the small un-resolved scales.
In all RANS turbulence models based on Boussinesq's hypothesis one tries to calculate a characteristic mixing length and a characteristic mixing velocity. Transport equations can be derived to obtain these to properties. They in turn contain unknown terms that have to be modelled.
Second order closure (first order closure means that one model the Reynold stress tensor directly, second order closure means that one derives transport equations for the Reynolds stress tensor and model the unknown terms in those equations).
In LES one resolve the large scales (eddies). The large scales carry information about the geometry/boundary conditions and change from flow to flow while the small scales are more universal and hence more suited for general modelling. All LES computations are three-dimensional and time dependent. LES require large computer resources.

VII. Terminology

Algebraic model In an algebraic turbulence model the Reynolds stress tensor is modelled through an algebraic relation (cf. differential model).
Boussinesq hypothesis The Boussinesq eddy-viscosity hypothesis assumes that the principal axes of the Reynolds stress tensor, τ_{ij}, are coincident with those of the mean strain-rate tensor, S_{ij}, at all points in a turbulent flow. The coefficient of proportionality between τ_{ij} and S_{ij} is the eddy viscosity, μ_T.
Continuum If the density of elements is high enough, we can approximate the system of interacting elements as a continuum. This expresses that their mutual interacting dominates over the individual motions, although these are not suppressed. See also, Knudsen number.
Differential model In a differential turbulence model the Reynolds stress tensor or some properties used to make up the eddy-viscosity are modelled through a differential relation e.g. a transport equation.
Diffusion (molecular) Transport of a property due to random molecular motion.
Diffusion (turbulent) Transport of a property due to turbulent motion.
Dissipation Transfer of kinetic energy to heat.
Eddy Two-dimensional vortex.
Eddy viscosity (turbulent viscosity) Turbulent "viscosity" in analogy with molecular viscosity. That the momentum flux, due to turbulent motion can be written as:

$$\mu_T = \tfrac{1}{2}\rho v_{mix} l_{mix}$$

Energy spectra Kinetic energy as a function of the wave number.
Flux The specific flow of a property through a boundary.
Homogenous turbulence The fluctuations are space independent, that is $u(x_o,y_o,z_o)=u(x,y,z)$.
Inertial sub-range A range of eddy sizes which are not directly affected by the energy maintenance and dissipation mechanisms.

Isotropic turbulence The fluctuations are the same in all directions, that is $u=v=w$.
Integral scale The integral scale is comparable with the largest scale in a flow. See definition of the integral scale in Pope p. 77.
Knudsen number A ratio between the mean free path and Kolmogorov micro-scale of length. The Knudsen number should be much less than 1 for the assumption of a continuum to be valid.
Kolmogorov micro-scale The smallest scales in the flow.
Mixing-length model That the momentum flux due to turbulent motion can be modelled using an appropriate length scale (the mixing-length).
Momentum Mass times velocity. Momentum is a conservative property. Density times velocity gives momentum per unit volume.
Reynolds number Describes the ratio between convection and molecular diffusion.
Skin friction The wall shear stress non-dimensionalised with the dynamic pressure.
Taylor micro-scale Can be viewed as an equivalent scale of the turbulent eddies
Transition When a flow changes from a laminar to a turbulent state one says that the flow undergoes transition. The change is due to growth of instabilities.
Vortex/Vortices Three-dimensional eddy/eddies
Wave number A function can be decomposed into waves of different periods or wave-lengths. The wave number is simply 2π divided by the wave-length.