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Numerical solution of a heat exchanger problem

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ABSTRACT

Nowadays, heat exchangers can be found everywhere: In heaters, in fridges, in boilers or in condensers of steam turbines. In all of these machines, the heat exchanger is a key element.

In a heat exchanger, heat is transported between two working fluids. These fluids aren't either in contact with each other or mixed. Usually they are separated by pipes or walls.

In this report, a typical heat exchanger problem is presented and shown how it can be solved with a numerical approach.

NOMENCLATURE

A	surface area of the heat exchanger, m^2
$\dot{C}$	heat capacity flux, W/K
$\dot{Q}$	heat flux, W
T	temperature, K
ΔT	temperature difference, K
c	specific heat, J/kgK
k	heat transfer coefficient, W/m²K
$\dot{m}$	mass flux, kg/s

Subscripts

h	hot fluid
c	cold fluid

Superscripts

‘	outlet

INTRODUCTION

Fig. 1 shows a counter flow heat exchanger. While the “cold” medium flows from the right-hand side to the left-hand side, the “hot” fluid flows from the left to the right.

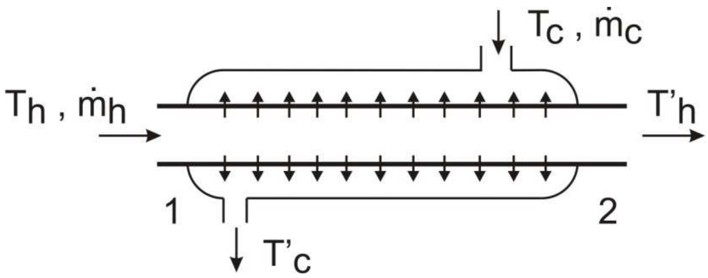

Fig. 1: counter flow heat exchanger

Fig. 2 shows the temperature profile of both fluids.

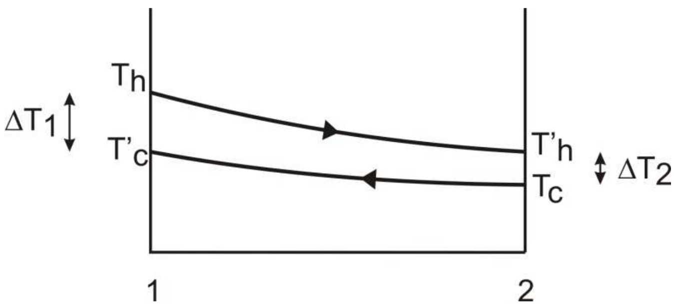

Fig. 2: Temperature profile

Usually, the inlet temperatures (T_h and T_c) are known, as well as the mass flux, the heat transfer coefficient and the surface area. Now the interesting question is: How to calculate the unknown values $\dot{Q}$, T_c’ and T_h’?

PROBLEM STATEMENT

The first law of thermodynamic provides the energy balance of the complete system and can be written as

$$\dot{Q} = (T_h - T_h^{'}) \cdot \dot{C}_h = (T_c - T_c^{'}) \cdot \dot{C}_c \quad (1)$$

where

$\dot{C}_h = \dot{m}_h \cdot c_h$ and $\dot{C}_c = \dot{m}_c \cdot c_c$

If supposed that the inlet temperatures (T_h / T_c) are known, there are still three unknown values: the outlet temperatures

(T'_h / T'_c) and the heat flux $\dot{Q}$. Therefore, a differential element is used (see Fig. 3).

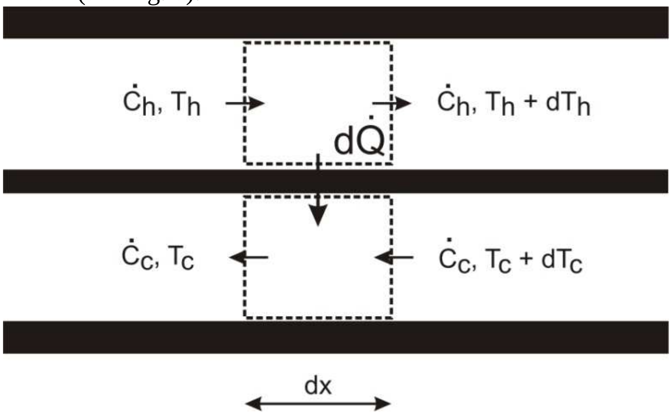

Fig. 3: Energy fluxes within two differential volume elements in a counter flow heat exchanger

The energy balance for the heat, which is absorbed or released by the two streams, is:

$$d\dot{Q} \subset = -\dot{C}_h \cdot dT_h = -\dot{C}_c \cdot dT_c \tag{2}$$

Furthermore, the following equation can also be used:

$$d\dot{Q} = k \cdot dA \cdot \Delta T \tag{3}$$

The heat transfer coefficient k is considered as a constant value all over the heat exchanger. In equation (3), ΔT is defined as T_h - T_c. But note: These temperatures are local temperatures, not the inlet temperatures.

With equation (3), equation (2) can be written as:

$$d\Delta T = dT_h - dT_c = d\dot{Q}\left(\frac{1}{\dot{C}_c} - \frac{1}{\dot{C}_h}\right) \tag{4}$$

If equation (3) and equation (4) are combined, one can form a differential equation out of these:

$$d\Delta T = dT_h - dT_c = d\dot{Q}\left(\frac{1}{\dot{C}_c} - \frac{1}{\dot{C}_h}\right) =$$

$$k \cdot dA \cdot \Delta T \cdot \left(\frac{1}{\dot{C}_c} - \frac{1}{\dot{C}_h}\right)$$

$$\Rightarrow \frac{d\Delta T}{\Delta T} = k \cdot dA \cdot \left(\frac{1}{\dot{C}_c} - \frac{1}{\dot{C}_h}\right) \tag{5}$$

The differential equation (5) can be solved by integration from point "1" to point "2" (see Fig. 1)

The solution is:

$$\ln\left(\frac{\Delta T_2}{\Delta T_1}\right) = -k \cdot A\left(\frac{1}{\dot{C}_h} - \frac{1}{\dot{C}_c}\right) \tag{6}$$

The term $\frac{1}{\dot{C}_c} - \frac{1}{\dot{C}_h}$ can be transformed with equation (1):

$$\left(\frac{1}{\dot{C}_h} - \frac{1}{\dot{C}_c}\right) = \frac{1}{\dot{Q}}(T_h - T_h' - T_c' + T_c) = \frac{1}{\dot{Q}}[(T_h - T_c') - (T_h' - T_c)] =$$

$$\frac{1}{\dot{Q}}(\Delta T_1 - \Delta T_2) \tag{7}$$

If equation (7) is applied to equation (6), one gets the following equation:

$$\dot{Q} = kA\frac{\Delta T_2 - \Delta T_1}{\ln\left(\Delta T_2 / \Delta T_1\right)} = kA\frac{[(T_h' - T_c) - (T_h - T_c')]}{\ln\left(\frac{T_h' - T_c}{T_h - T_c'}\right)} \tag{8}$$

Equation (8) still contains 3 unknown values. So, it's still very hard to solve this equation. However, it can be further rearranged if equation (1) is used two times. From equation (1), one can get equation (9) and (10):

$$T_h' = T_h - \frac{\dot{C}_c}{\dot{C}_h}(T_c' - T_c) \tag{9}$$

and

$$\dot{Q} = (T_c' - T_c) \cdot \dot{C}_c \tag{10}$$

With equation (9), the right hand side of equation (8) can be written as:

$$kA\frac{[(T_h' - T_c) - (T_h - T_c')]}{\ln\left(\frac{T_h' - T_c}{T_h - T_c'}\right)} = kA\frac{\left[(T_h + \frac{\dot{C}_c}{\dot{C}_h}(T_c - T_c') - T_c) - T_h + T_c'\right]}{\ln\left(\frac{T_h + \frac{\dot{C}_c}{\dot{C}_h}(T_c - T_c') - T_c}{T_h - T_c'}\right)} =$$

$$= kA\frac{\left[\frac{\dot{C}_c}{\dot{C}_h}(T_c - T_c') - T_c + T_c'\right]}{\ln\left(\frac{T_h + \frac{\dot{C}_c}{\dot{C}_h}(T_c - T_c') - T_c}{T_h - T_c'}\right)}$$

When this and equation (10) is applied to equation (8), the equation can be written as:

$$(T_c' - T_c) \cdot \dot{C}_c = kA\frac{\left[\frac{\dot{C}_c}{\dot{C}_h}(T_c - T_c') - T_c + T_c'\right]}{\ln\left(\frac{T_h - \frac{\dot{C}_c}{\dot{C}_h}(T_c' - T_c) - T_c}{T_h - T_c'}\right)} \tag{11}$$

Since T_c, T_h, C_c and C_h are known, equation (11) has only one unknown value. But unfortunately the equation can't be solved by hand since it's not possible to write it in this form:

T'_c = …

In order to solve this equation anyway, a numerical approach is chosen. This approach is shown in the next chapters.

SOLUTION OF THE PROBLEM

In order to get a specific value for T_c' from equation (11), one can solve this problem graphically. Since both side of equation (11) can be used to calculate $\dot{Q}$, $\dot{Q}$ can be plotted as a function of T_c', where T_c' is a set of values between 283K (= T_c) and 363K (= T_h). Furthermore, the following values are considered as known:

T_c = 283K
T_h = 363K
A = 2,1 m²
$\dot{C}_c = 6300\,W/K$
$\dot{C}_h = 8400\,W/K$

The plot of the two functions looks like this:

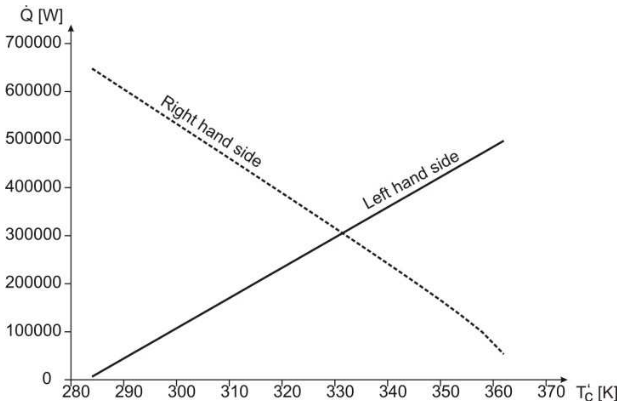

Fig. 4: Plot of the two side of equation (11)

Fig. 4 shows the plot of both functions. At the point where both function cross each other, there is the sought-after temperature T_c'. With a ruler, it's very easy to get a solution from this figure. It's between 330K and 340K. But this solution is not very accurate. Therefore, a little algorithm is developed. With this algorithm, special software can be used to calculate an exacter solution for T_c'. The software, used for this problem is called "Scilab".

THE ALGORITHM

Equation (11) is transformed once more and looks like this:

$$f(T_c') = (T_c' - T_c)\cdot\dot{C}_c - kA\frac{\left[\frac{\dot{C}_c}{\dot{C}_h}(T_c - T_c') - T_c + T_c'\right]}{\ln\left(\frac{T_h + \frac{\dot{C}_c}{\dot{C}_h}(T_c - T_c') - T_c}{T_h - T_c'}\right)} \quad (12)$$

Now, the algorithm will search for a solution of Equation (12) until f(T'$_c$) = 0. But since processors can't handle infinite small numbers, a break criterion is introduced: If the absolute value of the f(T'$_c$) is smaller than 10^{-3}, then the algorithm considers this as f(T'$_c$)=0 and stops the calculation process.

As a starting point, T'$_c$ is set to 285K. Then the difference between both sides is calculated. One can see that the difference is bigger than 10^{-3} (see Fig. 4). Therefore, T'$_c$ is increased by 10K (=ΔT) and the difference is calculated again. This procedure is repeated until the algebraic sign of the difference changes. If this happens, the algorithm "stepped over" the solution to the equation. Therefore, ΔT is subducted, divided by two and then added to T'$_c$ again. And the calculation starts again. If the algebraic sign changes again, ΔT is subducted and divided again.

To ensure that this procedure has a finite number of steps, a second break criterion is introduced: ΔT has to be higher than 10^{-7}. If it is smaller, then the algorithm stops and T'$_c$ is considered as found.

So, basically, the algorithm starts form one side of the crossing point and tries to approximate it as near as possible. If it steps over the crossing point (because ΔT is too big), it steps back and tries a smaller ΔT and so on. Fig. 5 shows the whole procedure graphically

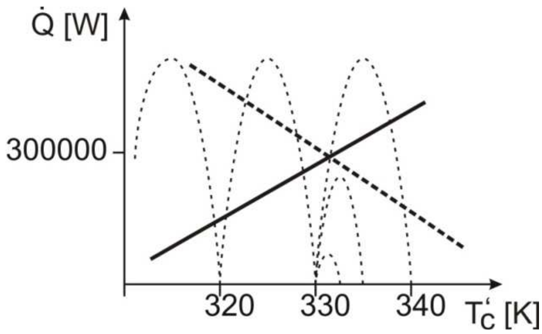

Fig. 5: Search for the crossing point. One can see a part of Fig. 4. The dotted line symbolizes the search procedure

IMPLEMENTION IN SCILAB

Scilab is a free numerical computational package, similar to the commercial software Matlab.

To show how the algorithm is realized in the software, Fig. 6 shows a structural diagram, which shows all the necessary steps. The source code for the algorithm can be found in the appendix.

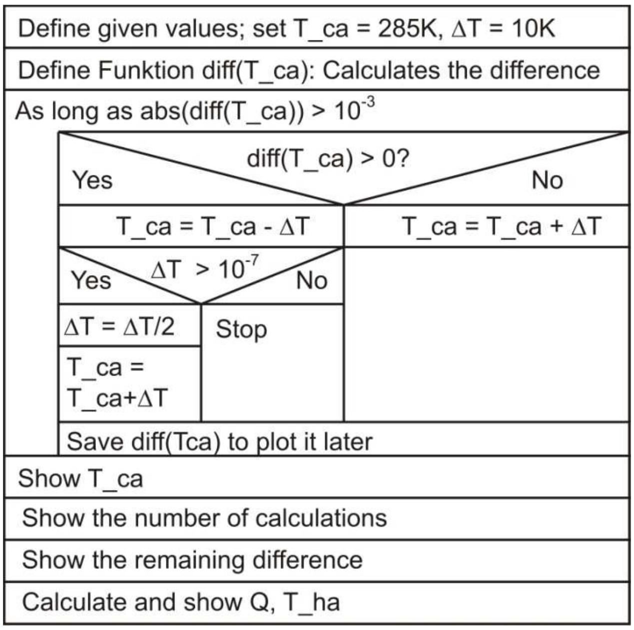

Fig. 6: structural diagram of the algorithm

SOLUTION

For this problem, T'_c is 331,60578 K, T'_h is 326.80816 K and the heat flux $\dot{Q}$ equals 305271,44 W (calculated with equation (10)).

These values are very accurate because if the break criteria were set to much smaller values, e.g. $\Delta T > 10^{-9}$ and diff(T_ca) > 10^{-5}, the values wouldn't really change.

An interesting question is how many calculation steps are required to find a solution. Fig. 7 shows the profile of the difference of the two equations (in logarithmic scale) over the number of calculated steps. The thick horizontal line represents the break criterion of diff(T_ca) > 10^{-3}. One can see, that after 43 iterations, a solution for T_ca is found (indicated by diff(T_ca)< 10^{-3}).

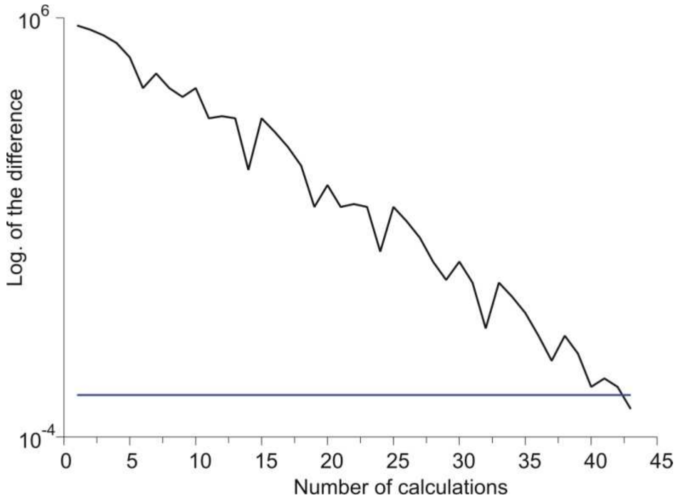

Fig. 7: profile of the difference over the number of calculations

CONCLUSIONS

As one can see, this algorithm is an easy, but very effective way to solve problems like this. This type of algorithm can be used for any kinds of problems when it comes to equations, which can't be solved by rearranging but with graphical analysis.

REFERENCES

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APPENDIX

The source code in Scilab (explanations are written in a different type of font):

```
xbasc();
clc;
clear;

Tce=283.15;                  //given values
Th=363.15;
A=2.1;
k=3900;
Cc=6300;
Ch=8400;

Tca=Tce+0.001;               //Starting point
deltaT=10;
i=0;

//Definition of a function to calculate a value with equation 11
function y=diff(T)
  y=(T-Tce)*Cc-k*A*((Cc/Ch)*(Tce-T)-
Tce+T)/log((Th+(Cc/Ch)*(Tce-T)-Tce)/(Th-T));
endfunction

//Loop to slove the equation step by step
while abs(diff(Tca))>10^(-3)& deltaT>10^(-7)
  i=i+1;
//the difference is saved so it can be plotted later

  difference(i)=diff(Tca);

//if the algebraic sign if the difference changes, then the following loop works
  if diff(Tca)>0
    Tca=Tca-deltaT;
// minimizing the step
    deltaT=deltaT/2;

  end

// next temperature step
  Tca=Tca+deltaT;
end

  i=i+1;
  difference(i)=diff(Tca);

//Output of the temperature, the difference and deltaT
  Difference=difference(i)
  deltaT
  Tca
  Q=(Tca-Tce)*Cc
  T_ha = (-1)*(Q-Th*Ch)/Ch

//plot of the number of calculations of the Difference between the two functions (in a logarithmic scale)
  x=zeros(i);
  for j=1:i
  x(j)=10^(-3);
  end

 //plot of the difference and a line x=10^-5 (=break criterion)

  plot2d("enl",[abs(difference),x])
  xlabel('Number of calculations');
  ylabel('Log. of the difference');
```